Disordered systems and Burgers' turbulence

Marc Mézard

Laboratoire de Physique Théorique de l'Ecole Normale Supérieure
(Unité propre du CNRS, associée à l'ENS et à l'Université de Paris Sud)
24 rue Lhomond, 75231 Paris Cedex 05, FRANCE
email: mezard@physique.ens.fr.

Abstract

The problem of fully developed turbulence is to characterize the statistical properties of the velocity field of a stirred fluid described by Navier stokes equations. The simplest scaling approach, due to Kolmogorov in 1941, gives a reasonable starting point, but it must be corrected due to the failure of naive scaling giving 'intermittency' corrections which are presumably associated with the existence of large scale structures. These scaling and intermittency properties can be studied analytically for the case of stirred Burgers turbulence, a kind of simplified version of Navier Stokes equations. We use the mapping between Burgers' equation and the problem of a directed polymer in a random medium in order to study the fully developed turbulence in the d dimensional forced Burgers' equation, in the limit of large dimensions. The stirring force corresponds to a quenched (spatio temporal) random potential for the polymer. A replica symmetry breaking solution of the polymer problem provides the full probability distribution of the velocity difference $u(r)$ between points separated by a distance r much smaller than the correlation length of the forcing. This exhibits a very strong intermittency which is related to regions of shock waves, in the fluid, and to the existence of metastable states in the directed polymer problem. We also mention some recent computations on the finite dimensional problem, based on various analytical approaches (instantons, operator product expansion, mapping to directed polymers), as well as a conjecture on the relevance of Burgers equation (with the length scale playing the role of time) for the description of the functional renormalisation group flow for the effective pinning potential of a manifold pinned by impurities. Preprint LPTENS 97/66.

1 Introduction

The understanding of fully developed turbulence is a well posed problem of mathematical physics [1]. It has been around for more than fifty years but in spite of many interesting developments very little is known for sure. In such a situation, it is clear that this field would benefit a lot from the existence of a solvable model which, even if not realistic, would exhibit the supposed properties of turbulence such as scaling and intermittency, and allow for a detailed study of them. The same role was played in the field of phase transitions by the Ising model and its solution by Onsager, hence the quest for an "Ising model of turbulence". Two candidates which have received a lot of attention recently are the stirred Burgers equation on the one hand, and the diffusion of a passive scalar in a random -gaussian- velocity field [2] on the other hand. I shall describe the former, focusing onto the solution of the infinite dimensional problem which we proposed with J.P. Bouchaud and G.Parisi [3]. Burgers turbulence displays very interesting relationship with a fundamental problem of the statistical mechanics of disordered systems, that of directed polymers in random media. Motivated both by the striking convergence between these two important fields and by my own background on disordered systems, I shall put particular emphasis on this relationship, which has already been quite useful so far, and will probably lead to other important developments in the next years.

The next section is a rapid presentation of the problem of fully developped turbulence. Sect.3 describes Burgers turbulence and its mapping onto a problem of directed polymers. Sect.4 contains a sketch of the solution in large dimensions. Sect.5 presents briefly some of the issues in the finite dimensional problem.

2   Fully developped turbulence

This short description is included here to set the stage for the next sections. It is very sketchy, I refer the reader to recent books on the subject for more precise presentations [1].

Consider a three-dimensional fluid, the velocity of which verifies Navier-Stokes equations, in the incompressible limit:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \ \vec{v} = \nu \nabla^2 \vec{v} + \vec{f}(\vec{x}, t) - \vec{\nabla}p \ \ ; \ \ \vec{\nabla}.\vec{v} = 0 \tag{1}$$

The coefficient ν is the viscosity. The force $\vec{f}$ is the external stirring force, which is supposed to inject energy into the system on a length scale Δ. Specifically, one can take for instance a gaussian distributed random force [1] characterized by its two moments:

$$< f^\mu(\vec{x}, t) >= 0 \ \ ; \ \ < f^\mu(\vec{x}, t) f^\nu(\vec{y}, t') >= \epsilon \delta(t - t') C^{\mu\nu}(|\vec{x} - \vec{y}|) \tag{2}$$

where the correlation function $C^{\mu\nu}(r)$ is normalised to one at the origin, and decays rapidly enough when r becomes larger or equal to Δ (I shall not enter here into the details of the tensorial structure). The parameter ϵ measures the energy injected into the fluid per unit time and unit volume.

The problem is to understand the statistical properties of the velocity field which is the solution of (1), in a regime which is both stationnary (i.e. neglecting initial transients) and isotropic (we assume periodic boundary conditions), in the limit of strong forcing. Ideally one might want to compute all moments such as $< v(\vec{x}_1, t_1) \ldots v(\vec{x}_q, t_q) >$, where the symbol $< . >$ means an average over various realizations of the random force. Here I shall focus for simplicity onto the moments of the velocity difference between two points, at equal times:

$$M_p =< |\vec{v}(\vec{x} + \vec{r}, t) - \vec{v}(\vec{x}, t)|^p > \tag{3}$$

or equivalently the probability distribution function (pdf) of $u = |\vec{v}(\vec{x} + \vec{r}, t) - \vec{v}(\vec{x}, t)|$.

This problem depends on three dimensionfull parameters, the viscosity ν with dimension $\frac{length^2}{time}$, the injection length scale δ, and the power injected per unit volume, ϵ, with dimension $\frac{length^2}{time^3}$. From these one can build one adimensional number, the Reynolds number: $Re = (\frac{\epsilon \Delta^4}{\nu^3})^{1/3}$. By "strong forcing" is meant the limit of large Re. Another important length scale is the dissipation length l_d which is the length below which the viscosity term becomes relevant: $l_d = \Delta/Re^{3/4}$.

The simplest description of what might happen in this problem is a scaling picture invented by Kolmogorov in 1941, usually called the "K41 theory" [4]. The energy is injected on a length scale Δ and dissipated on the much smaller length scales of order l_d. In between lies the so called inertial regime where the energy is transferred towards smaller and smaller length scales (a common metaphor is that of large eddies decaying into smaller ones). This is called the inertial regime and is supposed to be universal (independent on the details of the forcing for instance). The basic K41 hypothesis is that in this inertial regime $l_d << r << \Delta$, the velocity field scales with respect to the distance:

$$u = |\vec{v}(\vec{x} + \vec{r}, t) - \vec{v}(\vec{x}, t)| \sim r^\alpha \tag{4}$$

This scaling, denoted here with $\sim$, should be understood in law, meaning that the pdf of u depends on the distance r in the form $P(u) = \frac{1}{r^\alpha} \hat{P}(\frac{u}{r^\alpha})$, where the exponent α and the scaling function $\hat{P}$ are to be determined. Scaling of other moments of v assume a similar form, which basically amounts to

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[1] A different problem, related to deterministic chaos, concerns how a non random force becomes equivalent to a random one. I do not address this issue here.
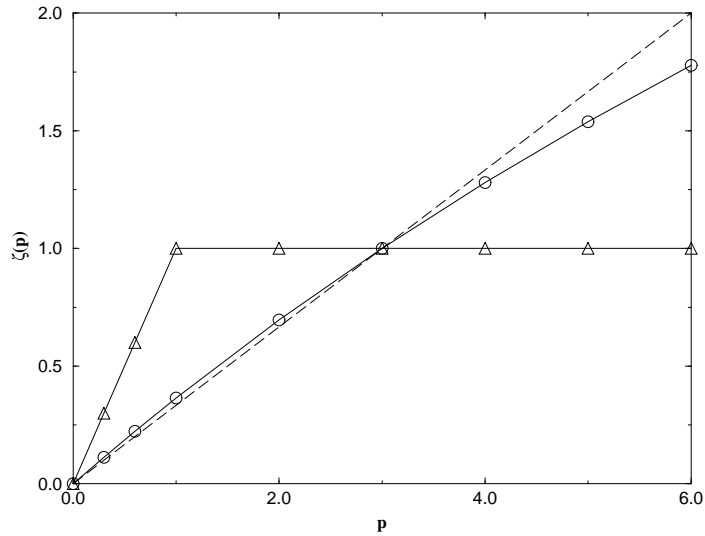
Figure 1: Sketch of the 'multifractal' spectrum $\zeta(p)$, giving the r dependence of the p th moment of the velocity field, both for the forced Burgers' equation (triangles) and hydrodynamical turbulence (circles). The dashed line is the K41 prediction.

saying that the instantaneous properties of the flow in the inertial regime are *statistically invariant* under a simultaneous rescaling of the lengths by a factor b and the velocities by the factor b^α.

One can guess the value of α through a dimensional argument which involves the following two hypotheses:

- Locality in scale space. The statistics of u depends on the separation r between the points, not on other length scales: $u \sim g(r, \nu, \epsilon)$.

- Existence of the zero viscosity limit. The statistics of u is well defined when $\nu \to 0$. Therefore the pdf of u depends only on r and ϵ, which implies, dimensionally: $u \sim (\epsilon r)^{1/3}$, that is $\alpha = 1/3$.

This K41 scaling, derived from two simple and reasonable assumptions, implies that the velocity difference between two points at distance r behaves as $r^{1/3}$ (Of course this holds only in the inertial range $\Delta >> r >> l_d$). Going to Fourier space, it is easy to show that the total kinetic energy contained in the Fourier modes with wave vectors $\vec{q}$ in the shell $|\vec{q}| \in [k, k + dk]$ scales as $k^{-5/3}dk$. Experimentally this scaling is very well verified at the level of the two point functions (i.e. M_2) [6]. Analytically one of the few sure results concerns the third moment of the u pdf, which is known to scale linearly [7]: $<u^3> = Cr$. On the other hand, a careful study of the higher order moments [5] favours a behaviour like $<u^p> = C_p r^{\zeta_p}$, where the exponent ζ_p is smaller than the value $p/3$ which would be obtained by K41, indicating a failure of simple scaling (see Fig. 1). This phenomenon has been given the name intermittency, because it is associated with the fact that the signal (e.g. the field u) has an intermittent structure both in space and in time, with bursts of activity separated by long quiet regions. This intermittency is related to the existence of large scale structure in the flow, such as vorticity filaments [8]. It suggests that the scaling is not uniform in space, but varies from point to point, a situation which can be described by a "multifractal" structure [9]. All these observations have prompted many interesting developments the description of which goes much beyond this presentation (see [1]).

As we see the situation is somewhat confusing. The most commonly accepted conjecture is that some scaling properties hold in the inertial regime, such as $<u^p> = C_p r^{\zeta_p}$, but that simple scaling does not hold: $\zeta_p \neq \alpha p$. If true, this is an interesting situation, and deriving these properties from the original stochastic partial differential equation (1) is a major challenge of theoretical physics. In recent years much attention has been paid to other "similar" but simpler problems, providing

an existence proof for intermittency together with some hints on its origin.

3 Burgers turbulence and directed polymers

The Burgers equation is defined in any dimension d (although most often it is studied in $d = 1$). The evolution equation for the velocity field is:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \, \vec{v} = \nu \nabla^2 \vec{v} + \vec{f}(\vec{x}, t) \;\; ; \;\; \vec{v} = -\vec{\nabla} h \tag{5}$$

It is a nonlinear stochastic partial differential equation which looks similar to the Navier Stokes one (1). The only, but crucial, difference is the second constraint: instead of an incompressible velocity, we consider the case where the velocity is a gradient field. For consistency the forcing is also supposed to be a gradient:

$$\vec{f} = -\vec{\nabla} \phi \tag{6}$$

Apart from this, the problem is defined as before: ϕ is supposed to be a gaussian process, white noise in time, with a certain spatial correlation length Δ, and we want to study the statistical properties of the flow at large Re. Because of the irrotational nature of Burgers flow the large scale excitations are very different from those of Navier Stokes. One should not consider this Burgers turbulence in any sense as an approximation to real turbulence, but as a different problem which we use as a laboratory for developing and testing theoretical ideas. (Another difference is the non conservation of energy by Burgers flow in $d > 1$; Surprisingly this does not seem to be crucial).

Clearly the scaling arguments of the previous section apply here as well, and the simple K41 scaling would predict again that $u = |\vec{v}(\vec{x} + \vec{r}, t) - \vec{v}(\vec{x}, t)| \sim r^{1/3}$. As we shall see this is too simple and there are strong intermittency effects.

The key ingredient at the heart of the solution of this problem is the existence of a non linear transformation which transforms it into a stochastic but linear problem, the "Hopf-Cole" transformation, which has been known for a long time [10]. Define $Z(\vec{x}, t)$ by $\vec{v} = -2\nu \vec{\nabla} \log Z(\vec{x}, t)$. It obeys the equation:

$$\frac{\partial Z}{\partial t} = \nu \nabla^2 Z + \frac{\phi}{2\nu} Z \tag{7}$$

This is a Schrödinger equation in imaginary time, for a particle moving in a time dependent random potential $\phi(\vec{x}, t)/(2\nu)$. It is natural to introduce the corresponding (Euclidean) path integral representation:

$$Z(\vec{x}, t) = \int d\vec{y}_0 \rho(\vec{y}_0) \int_{\vec{y}(0) = \vec{y}_0}^{\vec{y}(t) = \vec{x}} \mathcal{D}(\vec{y}) \exp \left(-\frac{1}{2\nu} \int_0^t d\tau \, \left(\frac{1}{2} \left[\frac{\partial \vec{y}}{\partial \tau} \right]^2 + \phi(\tau, \vec{y}(\tau)) \right) \right) \tag{8}$$

where the integral $\mathcal{D}(\vec{y})$ is on all the paths $\vec{y}(\tau)$ arriving at x for $\tau = t$, and the distribution $\rho(\vec{y}_0)$ encodes the initial velocity field (see Fig.2). In this form one can identify the partition function for a directed polymer in a random medium (DPRM). The directed polymer is characterised by the curve $\vec{y}(\tau)$. It lives an a $d+1$ dimensional space, which is the space-time of our original problem, and it is directed in the 'time' direction. Its energy is the sum of an elastic term, $E_{el} = \int d\tau \, \left(\frac{\partial \vec{y}}{\partial \tau} \right)^2$, and a potential energy $E_p = \int d\tau \, \phi(\tau, \vec{y}(\tau))$. This potential energy corresponds to a random pinning potential of the polymer. Identifying the temperature as $T = 2\nu$, we recognize in (8) that each configuration $\vec{y}(\tau)$ of the polymer is given a Boltzmann weight $\exp - \left(\frac{E_{el} + E_p}{T} \right)$. The function $Z(\vec{x}, t)$ is the partition function for the polymers constrained to arrive on the point $\vec{x}, t$. In order to get back to the Burgers velocity field, one must first compute the polymer's free energy $F(\vec{x}, t) = -2\nu \log Z(\vec{x}, t)$ and take its gradient:

$$\vec{v}(\vec{x}, t) = -2\nu \vec{\nabla} \log Z(\vec{x}, t) \tag{9}$$

It is important to recognise that in the polymer language the potential $\phi(\vec{x}, t)$ is a *quenched* random potential: we need to evaluate the free energy of the polymer in a fixed realisation of ϕ (a fixed
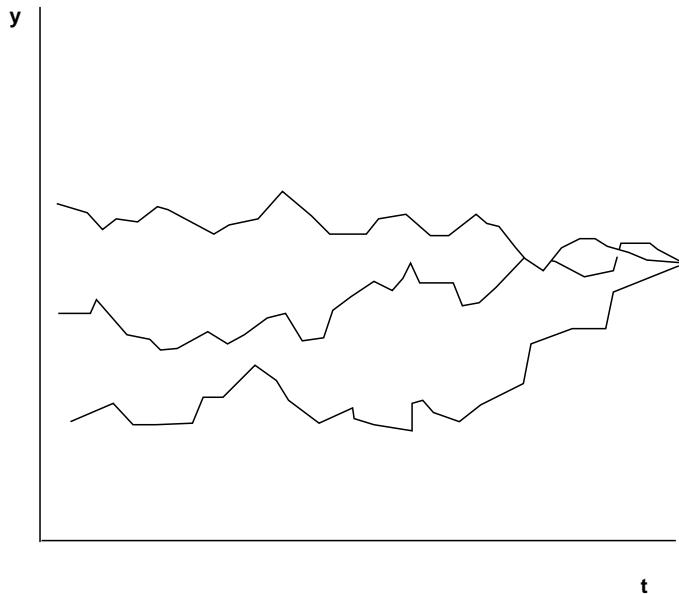
Figure 2: Directed polymers in random media: The partition function (8) is given by the sum over all directed paths $\vec{y}(\tau)$ arriving at a given point, each path being given a Boltzman weight depending on its elastic energy and the random pinning energy. The velocity in Burgers equation is proportional to the derivative of the polymer's free energy with respect to moving the arrival point at fixed time

sample), and then do the sample averaging. This is not intuitive: In the original problem ϕ is a time dependent field from which the stirring force derives. After the mapping it has to be considered as a quenched potential. This is because we have gone to a space-time description: if we consider the initial Burgers problem in dimension d, the space where the directed polymer lives is $d + 1$ dimensional.

Directed polymers in random media have received a lot of attention in recent years, both for their own interest and also for their relevance in the description of magnetic flux tubes in type II superconductors [11]. However these systems have a very important difference with respect to our problem issued from Burger's turbulence: in general the correlation of the random pinning potential are short range, and one is interested in what happens on length scales much larger than this correlation. In our problem we need to study the case where Δ is large, in the sense that the inertial regime takes place on length scales much shorter than Δ. This will prevent us to use directly the known results on the DPRM.

Yet let us mention briefly some of the properties which are known for the case of short range correlated disorder (this case can be studied also with a lattice discretization of the space-time, each point of the lattice being given an independent value of the potential). In this case, the statistical properties of DPRM also obey some scaling relations, and no sign of intermittency has been reported so far. For instance the lateral wandering of the polymer, far away from its extremities, is characterized by an exponent ζ defined by:

$$< E_T \left[\vec{y}(\tau) - \vec{y}(\tau'))^2 \right] >= C|\tau - \tau'|^{2\zeta} . \tag{10}$$

where $E_T[A]$ is the expectation value of the quantity A with the Boltzmann weight contained in (8), while $<>$ stands as before for the average over the random potential ϕ. Another exponent of interest is the exponent ω characterizing the fluctuations of the free energy when one varies the arrival point of the polymer:

$$< (F(\vec{x}, t) - F(\vec{x'}, t))^2 >= C|\vec{x} - \vec{x'}|^{2\omega/\zeta} \tag{11}$$

In order to get the velocity correlation function we need to differentiate this twice, and we thus get:

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$\alpha = \omega/\zeta - 1$. An important identity between the exponents is derived from Galilean invariance which imposes that the terms $\frac{\partial \vec{v}}{\partial t}$ and $(\vec{v} \cdot \vec{\nabla}) \, \vec{v}$ must have the same scaling behaviour, thus fixing [12]: $\omega = 2\zeta - 1$. The exponents are known exactly only in $1 + 1$ space-time dimension, where $\zeta = 2/3$. The general problem of the pinning of elastic manifolds (of arbitrary internal dimension and codimension) by random impurities is a major one in the statistical physics of disordered systems, but only approximate answers are known so far [11, 15].

We now close this parenthesis and get back to our problem where the random potential is correlated on a length scale Δ which is much larger than the (inertial) scale on which we want to study the system. A priori, this is a much simpler problem: the random potential varies slowly, and thus we should be allowed to substitute it by a linear potential. This is the essence of the random force approximation, pioneered in the study of the pinning of elastic manifolds by Larkin and his collaborators in the seventies [13]:

$$\phi(\tau, \vec{x}) \simeq A(\tau) - \vec{f}(\tau).\vec{x}(\tau) \tag{12}$$

In this case the total energy of the polymer is a quadratic function of the position, and one can solve the problem completely. Introducing the Fourier transforms $\vec{x}(\omega)$ and $\vec{f}(\omega)$ of the fields $\vec{x}(\tau)$ and $\vec{f}(\tau)$, we find immediately the thermal average of the polymer's position $\vec{x}$ (computed with the Boltzmann weight $(1/Z) \exp\left[-\int d\tau (\partial \vec{x}/\partial \tau)^2 + \int d\tau \vec{f}(\tau).\vec{x}(\tau)\right]$):

$$E_T(\vec{x}(\omega)) = \frac{\vec{f}(\omega)}{\omega^2} \tag{13}$$

Using the fact that $< \vec{f}(\omega).\vec{f}(\omega') > \propto \delta(\omega + \omega')$, one immediately gets:

$$< E_T\left([\vec{x}(\tau) - \vec{x}(\tau')]^2\right) > \propto \int \frac{d\omega}{\omega^4}[1 - \cos(\omega(\tau - \tau'))] \tag{14}$$

which seems to imply that the wandering exponent is $\zeta = 3/2$. This is a very interesting result since it leads to $\omega = 2$ and $\alpha = 1/3$. At first sight, we have derived $K41$ scaling from a two lines computation on the directed polymer, using only the random force approximation which should be valid in the inertial range. However this is wrong. The technical reason is obvious: the integral (14) over the frequency is divergent at small ω. The consequences are far reaching:
- The divergence means that it is impossible to decouple the small scales from the large ones: the random force approximation is never valid.
- We must thus study a DPRM in the presence of a full random potential, not in a random force field. A crucial difference is that in presence of a random potential there are many metastable states, while there is no metastability in the random force approximation.
- As we shall see, the existence of metastable states of the potential is intimately related to the breakdown of simple scaling, and the appearance of intermittency.

4   The solution of forced Burgers turbulence in large dimensions

So we face the difficult problem of computing the properties of a DPRM in the presence of a random pinning *potential*, correlated on a scale Δ. This may not look much easier than our original Burgers problem, but it turns out that some of the powerful methods developed in the study of disordered systems can be brought to bear on the DPRM. Technically this becomes rather involved, I shall just give the main steps of the computation.

The first step uses the replica method [14]. One introduces the partition function Z^n for n replicas of the directed polymer, and averages it over the distribution of the random potential. The analytic continuation to $n \to 0$ will thus provide the average free energy, which we seek. The average velocity correlations will be obtained by a similar procedure [3]. The average of Z^n is:

$$< Z^n >= \prod_{a=1}^{n} \int \mathcal{D}(\vec{y}_a)e^{-H_n} \tag{15}$$

where the average over disorder has introduced an effective *attraction* between the various lines:

$$H_n = \int_0^t d\tau \left[\frac{c}{2} \sum_{a=1}^n \left(\frac{d\vec{y}_a}{d\tau} \right)^2 + + \sum_{a,b=1}^n C(\vec{y}_a(\tau) - \vec{y}_b(\tau)) \right] \tag{16}$$

where C is the correlation function of the random pinning potential. In this language we must study either the statistical mechanics of n lines or the Euclidean quantum mechanics of n particles, interacting by pairs through the potential C (which is attractive and has a range equal to Δ). We should compute the Green function for this problem and then continue it analytically for $n \to 0$. So far this can be done exactly only in two cases: the case of one dimension and $\Delta = 0$ (giving back the exponent $\zeta = 2/3$ of usual DPRMs), or the infinite dimensional case, in which the variational method developped below becomes exact.

To study this problem of n interacting elastic lines, we have used a variational approach. It finds the best approximation to the system by a quadratic Hamiltonian, belonging to the family:

$$H_v = \frac{1}{2} \sum_{a,b} \int d\tau \int d\tau' K_{ab}(\tau - \tau') \vec{x}_a(\tau).\vec{x}_b(\tau') \tag{17}$$

The idea is to compute the variational free energy, which is a functional of $K_{ab}(\tau)$, and then to find the best set of functions $K_{ab}(\tau)$ which optimizes the free energy. This variational method is a Hartree like approximation, which becomes exact in the limit of large dimensions. It has been used a lot for the pinning of random manifolds [15], where it is able to give information on the phase diagram and the exponents in the glassy phase (the phase with many metastable states). A priori, the problem is symmetric under the permutation of the replica indices. What is found here is that, if the Reynolds number of the initial Burgers problem is large enough (i.e. the pinning potential is strong enough), there is a spontaneous breakdown of this permutation symmetry: In the optimal solution, the various functions K_{ab}, $a \neq b$, are not all equal. This phenomenon, called replica symmetry breaking (RSB), has been studied at length in the spin glass problem [14]. Finding a consistent scheme for RSB has been a major achievement in the mean field model of spin glasses [16]. Here we have used the same scheme of RSB in our directed polymer problem. We shall not describe it since it would take us too far (see [3]). Instead we shall state the results of the variational method with RSB in physical terms. Intuitively, it is enough to think of the RSB effect as being associated with the existence of many metastable states: then "different replicas may fall into different states"...

The physical description of the RSB solution is best described by the procedure which generates a velocity pattern $\vec{v}(\vec{x}, t)$ at a fixed time t, for one given realization of the forcing. This procedure is as follows:
- One chooses $M(\gg 1)$ points $\vec{r}_\alpha$ independently, with a uniform distribution in the box.
- For each point, one chooses one "free energy" f_α. These free energies are independent, taken from a Poisson process of density $\exp(f)$.

One then generates the partition function as a weighted sum of Gaussians:

$$Z(\vec{x}) = \sum_\alpha \exp \left(-Re \left[f_\alpha + \frac{(\vec{x} - \vec{r}_\alpha)^2}{2\Delta^2} \right] \right) \tag{18}$$

and the velocity is given by the Hopf Cole transformation (9). The number of states is irrelevant, as long as it is large, because of the exponential distribution of the free energies, which ensures that the distribution of the gaps (differences between the smaller free energies) is M independent. For large Reynolds numbers, the sum over the various points α is dominated, for each value of $\vec{x}$, by one α which we call $\alpha^*(\vec{x})$. This dominant $\alpha^*(\vec{x})$ is locally independent of $\vec{x}$ (when $Re \to \infty$), but it jumps from time to time when one varies $\vec{x}$. Therefore, the structure of the flow is locally radial:

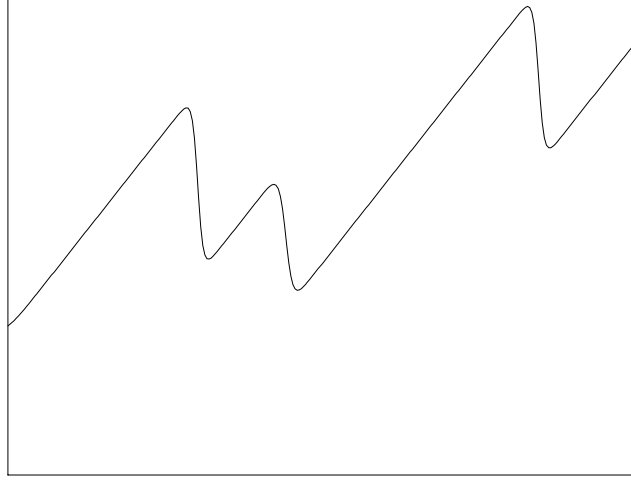$$\vec{v}(\vec{x}) = v_\Delta \frac{\vec{x} - \vec{r}_{\alpha^*(\vec{x})}}{\Delta} \tag{19}$$

Figure 3: Schematic structure of Burgers flow in one dimension, obtained by the replica variational approach to the DPRM formulation (18). The picture gives the velocity $v(x,t)$ as a function of x for a given time t. The typical width of the cells is of the order of the injection length scale Δ, while the width of the shocks behaves as Δ/Re. The structure is similar to that of decaying Burgers turbulence, and leads to an extreme type of intermittency effect(20,22).

The flow organises itself into cells of typical size of order the injection scale Δ, with a discontinuity of $\vec{v}$ (a 'shock') between cells. From the representation (18) one immediately sees that, for finite Re, the region of the shock has a size of order Δ/Re. A typical structure of the flow in one dimension is shown in Fig.3. In higher dimensions, the regions of shocks separating the cells where the flow is radial have dimension $d-1$.

From this physical description of the replica solution one can infer the scaling exponents. For simplicity the discussion is presented in the $d=1$ case. Consider the velocity difference at distance r, $u = |v(x+r,t) - v(x,t)|$, in the inertial range $\Delta/Re << r << \Delta$.
- If there is no shock in the range $[x, x+r]$, then $u = v_\Delta r/\Delta$.
- If there is a shock, then u is a random variable of typical size of order v_Δ. As the probability of there being a shock is of order r/Δ, one expects that the pdf of u will take the following form:

$$P(u) \sim (1 - \frac{r}{\Delta})\delta\left(u - v_\Delta \frac{r}{\Delta}\right) + \frac{r}{\Delta}f(u) \tag{20}$$

This is indeed confirmed by the detailed replica computation, which leads, in the limit of large Re, to a complicated expression [3, 17], the structure of which indeed has the form (20). Let us now estimate the various moments of u using expression (20):

$$\overline{u^p} \sim (1 - \frac{r}{\Delta})(\frac{v_\Delta r}{\Delta})^p + \frac{r}{\Delta} A_p , \tag{21}$$

where $A_p = \int du\, u^p f(u)$. In the inertial range, one finds that for *all* p larger than 1, one has

$$\overline{u^p} = A_p v_\Delta^p \frac{r}{\Delta} , \tag{22}$$

whereas for $p < 1$, one finds

$$\overline{u^p} \propto v_\Delta^p (\frac{r}{\Delta})^p . \tag{23}$$

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So we face an extreme case of intermittency where all the exponents ζ_p, with $p > 1$, are equal to one (see Fig.1). Notice that this intermittency has been found here for stirred Burgers turbulence. The solution, which is based on a variational method, is exact only in the limit of large dimensionality. It is interesting to notice however that the same spectrum of exponents ζ_p is found in *decaying* Burgers turbulence in one dimension [18], and has been called a 'bifractal spectrum' (with reference to the multifractal description of intermittency [9]). In fact there exists a much deeper corresondence, since our exact result for $P(u)$ in large dimensions and stirred turbulence is identical (up to rescalings of lengths and velocities) is identical to the one obtained by Kida in the decaying Burgers turbulence in $d = 1$, in the case where the initial velocity field is the gradient of a gaussian random field with local correlations [19]. This is not fortuitous, and is in fact due to the universality of extreme event statistics. This universality, which leads to an exponential ('Gumbel') distribution of low lying states, is probably at the heart of the success of the replica symmetry breaking in many situations, since the rsb is well known to describe exactly such a distribution [17, 20].

5   The finite dimensional case

A very natural question is: what survives of this large d solution in the finite dimensional case? Actually the stirred Burgers turbulence in the finite d case, and particularly the case $d = 1$, has been the subject of an intense activity in the recent years [22, 23, 24, 25, 26]. The situation is still not totally clarified, although interesting progress has been made. I shall not try to review all the various methods which have been used, but just point out briefly a few facts in relation to our previous picture. At large Reynolds, the instantaneous velocity field is a succession of regions where the velocity is continuous, separated by shocks [25]. The pdf of the velocity difference, u, has a 'right tail' (tail at positive u) which is dominated by the smooth regions. In these smooth regions one can locally expand the velocity as

$$v(x,t) \sim \lambda(t)(x - x_0(t)) \tag{24}$$

and the slope λ satisfies a Langevin type equation:

$$\frac{d\lambda}{dt} = -\lambda^2 + \eta \tag{25}$$

(where η is a noise), from which one deduces immediately [26] a right tail of the pdf of the slopes behaving as $\exp(-C\lambda^3)$ and therefore a right tail of $P(u)$ behaving as $\exp(-C'u^3)$. This result was actually first found by Polyakov [23] using a totally different method: He writes the Hopf equation for the generating function $< \exp(\mu(v(x + r) - v(x))) >$. This Hopf equation contains an anomalous term of the type $\nu < v''(x) \exp(\mu(v(x + r) - v(x))) >$ which has to be dealt with, and for which a certain form of operator product expansion has been proposed. This leads to the above right tail of $P(u)$. However the same right tail is found independently of the detailed form which is supposed for the operator product expansion, for which it does not provide a real test.

The presence of this right tail in $\exp(-C'u^3)$ has been confirmed by several other approaches. One appealing formulation is the instanton approach of Gurarie and Migdal[24]. The procedure consists in writing the dynamical field theory for the Burgers flow, and seeking the instanton configuration of the velocity field, and the field conjugate to it, which provide the tail of the pdf. In principle, such a procedure is allowed whenever one is interested in some rare events. Here one fixes a value of u which is improbably large, and one basically seeks the most probable set of velocities and stirring which lead to this value. It can also be extended to higher dimensions where it leads to a similar right tail of $P(u) \sim \exp(-C'u^3)$, but the constant C' turns out to be proportional to the dimension d. This is the reason why such a tail is not seen in the (large d) solution of the previous sections. It also shows that the large u limit and the large d limit do not always commute.

A much more complicated problem is the form of the pdf $P(u)$ at negative u. The reason of the complication is clear: this negative u region is dominated by shocks, and understanding it requires a control of the statistics of the shocks. While this is doable in the case of decaying Burgers

turbulence where the shocks just merge, it is much more complicated in the forced case where new shocks keep appearing at all times. So far the situation is not totally clear and some conflicting predictions have been made on the left tail, which I shall not try to review.

6   Concluding remarks

As mentioned above, I think that one virtue of the large d solution is to provide an existence proof for an intermittency situation generated by a non linear partial differential equation with a structure having some similarities to that of Navier Stokes. Furthermore, the intermittency property is clearly associated with the existence of large scale structures, the shocks, and it can be studied in many details.

To conclude I would like to point out some rather speculative idea which is another interesting relation between disordered systems and Burgers turbulence, going in the reverse direction. If one considers the pinning of elastic manifolds by a random potential, there exist at the moment only two quantitative approaches to the study of this problem starting from a microscopic description. One is the replica variational method mentionned above [15], the other one is the functional renormalization group developed by D. Fisher and his collaborators [27]. It has been understood recently that both approaches suggest the same picture for the effective free energy landscape of these manifolds at large scale (at least for manifolds of internal dimension close to 4, embedded into a space of large dimension): it is given by a succession of parabolic wells of random depth, matching on singular points where the effective force is discontinuous. These parabolas are themselves subdivided into smaller parabolas, corresponding to the motion of smaller length scales, in a hierarchical manner [28]. Some preliminary investigation [29] points towards the fact that the renormalisation group flow for the gradient of the effective pinning potential is actually described by a stirred Burgers-like equation (although the non linear term is superficially irrelevant, it may become important because of the breakdown of scaling and the generation of shocks). This would naturally account for the above picture, since we know that a Burgers flow leads to a velocity pattern which is locally radial, associated thus with a locally parabolic effective pinning potential (free energy). If it would be true, it would give another interesting perspective to the studies of forced Burgers turbulence and the relevance of intermittency: the flow of the effective pinning potential would get away from the gaussian subspace (which is the only one studied so far), and one would rather need to project it onto a different subspace, that of locally parabolic functions, in order to get the correct fixed point and exponents.

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